

CALDERÓN PROBLEM FOR MAXWELL'S EQUATIONS IN TWO DIMENSIONS

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ABSTRACT. We prove the global uniqueness in determination of the conductivity, the permeability and the permittivity of two dimensional Maxwell's equations by partial Dirichlet-to-Neumann map limited to an arbitrary subboundary.

Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$, and $\vec{\nu} = (\nu_1, \nu_2, \nu_3) \in \mathbf{R}^3$ be a unit outward normal vector to $\partial\Omega$ and let $i = \sqrt{-1}$. Let $E = (E_1, E_2, E_3)$ be the electric field, $H = (H_1, H_2, H_3)$ the magnetic field, σ be the conductivity, μ the permeability and ϵ permittivity. In this paper we assume that $E(x), H(x), \sigma(x), \mu(x), \epsilon(x)$ are independent of the third component x_3 of $x = (x_1, x_2, x_3) \in \mathbf{R}^3$. Then we embed Ω in \mathbf{R}^2 and regard Ω as a domain in \mathbf{R}^2 . Then $\vec{\nu} = (\nu_1, \nu_2, 0)$ and

$$\vec{\nu} \times E = \begin{pmatrix} \nu_2 E_3 \\ -\nu_1 E_3 \\ \nu_1 E_2 - \nu_2 E_1 \end{pmatrix} \quad \text{on } \partial\Omega$$

and Maxwell's equations are given by

$$(0.1) \quad L_{1,\mu,\gamma}(x, D)(E, H) := \begin{pmatrix} \partial_{x_2} E_3 \\ -\partial_{x_1} E_3 \\ \partial_{x_1} E_2 - \partial_{x_2} E_1 \end{pmatrix} - i\omega\mu \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = 0, \quad \text{in } \Omega,$$

and

$$(0.2) \quad L_{2,\mu,\gamma}(x, D)(E, H) := \begin{pmatrix} \partial_{x_2} H_3 \\ -\partial_{x_1} H_3 \\ \partial_{x_1} H_2 - \partial_{x_2} H_1 \end{pmatrix} + i\omega\gamma \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = 0, \quad \text{in } \Omega.$$

Here and henceforth we set

$$\gamma = \epsilon + \frac{i\sigma}{\omega}$$

and

$$L_{\mu,\gamma}(x, D)(E, H) = (L_{1,\mu,\gamma}(x, D)(E, H), L_{2,\mu,\gamma}(x, D)(E, H)).$$

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Let $\tilde{\Gamma}$ be some fixed open subset of $\partial\Omega$ and $\Gamma_0 = \partial\Omega \setminus \tilde{\Gamma}$. Consider the following Dirichlet-to-Neumann map

$$(0.3) \quad \Lambda_{\mu,\gamma} f = \vec{\nu} \times H = \begin{pmatrix} \nu_2 H_3 \\ -\nu_1 H_3 \\ \nu_1 H_2 - \nu_2 H_1 \end{pmatrix} \quad \text{on } \tilde{\Gamma},$$

where

$$L_{\mu,\gamma}(x, D)(E, H) = 0 \quad \text{in } \Omega, \quad \vec{\nu} \times E|_{\Gamma_0} = 0, \quad \vec{\nu} \times E|_{\tilde{\Gamma}} = f.$$

In general for some values of the parameter ω , the boundary value problem

$$(0.4) \quad L_{\mu,\gamma}(x, D)(E, H) = 0 \quad \text{in } \Omega, \quad \vec{\nu} \times E|_{\Gamma_0} = 0, \quad \vec{\nu} \times E|_{\tilde{\Gamma}} = f$$

may not have a solution for some f . By $D_{\mu,\gamma}$ we denote the set of functions $f \in H^1(\tilde{\Gamma})$ such that there exists at least one solution to (0.4). As for the mathematical theory on the boundary value problem for Maxwell's equations, we refer for example to Dautray and Lions [3].

In general for some $f \in D_{\mu,\gamma}$, there exists more than one solutions. In that case as the value of $\Lambda_{\mu,\gamma} f$, we consider the set of all functions $\vec{\nu} \times H$ where the pairs (E, H) are the all possible solutions to (0.4). Thus our definition of the Dirichlet-to-Neumann map is different from the classical one, and we have to specify the conception of the equality of the Dirichlet-to-Neumann maps.

Definition. We say that the Dirichlet-to-Neumann maps Λ_{μ_1,γ_1} and Λ_{μ_2,γ_2} are equal if $D_{\mu_1,\gamma_1} \subset D_{\mu_2,\gamma_2}$ and for any pair (E, H) which solves

$$L_{\mu_1,\gamma_1}(x, D)(E, H) = 0 \quad \text{in } \Omega, \quad \vec{\nu} \times E|_{\Gamma_0} = 0, \quad \vec{\nu} \times E|_{\tilde{\Gamma}} = f,$$

there exists a pair (\tilde{E}, \tilde{H}) which solves

$$L_{\mu_1,\gamma_1}(x, D)(\tilde{E}, \tilde{H}) = 0 \quad \text{in } \Omega, \quad \vec{\nu} \times \tilde{E}|_{\Gamma_0} = 0, \quad \vec{\nu} \times \tilde{E}|_{\tilde{\Gamma}} = f$$

and

$$\vec{\nu} \times H = \vec{\nu} \times \tilde{H} \quad \text{on } \tilde{\Gamma}.$$

Then we can state our main result:

Theorem Let $\mu_j, \epsilon_j, \sigma_j \in C^5(\bar{\Omega})$ for $j \in \{1, 2\}$ and μ_j, ϵ_j be the positive functions on $\bar{\Omega}$. Suppose that $\Lambda_{\mu_1,\gamma_1} = \Lambda_{\mu_2,\gamma_2}$ and

$$\mu_1 - \mu_2 = \frac{\partial \mu_1}{\partial \nu} - \frac{\partial \mu_2}{\partial \nu} = \gamma_1 - \gamma_2 = \frac{\partial \gamma_1}{\partial \nu} - \frac{\partial \gamma_2}{\partial \nu} = 0 \quad \text{on } \tilde{\Gamma} \quad \text{and} \quad \frac{\partial(\sqrt{\gamma_1} - \sqrt{\gamma_2})}{\partial \nu} = 0 \quad \text{on } \Gamma_0.$$

Then $\mu_1 = \mu_2$ and $\gamma_1 = \gamma_2$.

See Caro, Ola and Salo [1] and Ola, Päiväranta and Somersalo [11] for the uniqueness results for Maxwell's equations in three dimensions. In the two dimensional case, we can reduce Maxwell's equations to the conductivity equation with zeroth order term and apply the uniqueness result in Imanuvilov, Uhlmann and Yamamoto [5] to prove the theorem.

Proof. First we observe that the system (0.1), (0.2) can be separated into the two independent systems of partial differential equations. The first system has the form

$$(0.5) \quad H_1 = \frac{1}{i\omega\mu}\partial_{x_2}E_3, \quad H_2 = -\frac{1}{i\omega\mu}\partial_{x_1}E_3, \quad \partial_{x_1}H_2 - \partial_{x_2}H_1 = -i\omega\gamma E_3 \quad \text{in } \Omega.$$

The second system can be written as

$$(0.6) \quad E_1 = -\frac{1}{i\omega\gamma}\partial_{x_2}H_3, \quad E_2 = \frac{1}{i\omega\gamma}\partial_{x_1}H_3, \quad \partial_{x_1}E_2 - \partial_{x_2}E_1 = i\omega\mu H_3 \quad \text{in } \Omega.$$

After we plug into the third equation of (0.5) expressions for H_1 and H_2 from the first two equations we obtain

$$(0.7) \quad L_{1,\mu,\gamma}(x, D)E_3 = \operatorname{div} \left(\frac{1}{i\omega\mu} \nabla E_3 \right) - i\omega\gamma E_3 = 0 \quad \text{in } \Omega.$$

Similarly, from (0.6) we obtain

$$(0.8) \quad L_{2,\gamma,\mu}(x, D)H_3 = -\operatorname{div} \left(\frac{1}{i\omega\gamma} \nabla H_3 \right) + i\omega\mu H_3 = 0 \quad \text{in } \Omega.$$

Observe that in order to solve the system (0.5) it suffices to solve the conductivity equation (0.7) and then determine the first two components of the magnetic field H by formulae

$$(0.9) \quad H_1 = \frac{1}{i\omega\mu}\partial_{x_2}E_3, \quad H_2 = -\frac{1}{i\omega\mu}\partial_{x_1}E_3.$$

Finally, using equation (0.9) in (0.7) to eliminate E_3 we obtain the last equation in (0.5).

Similarly in order to construct solution to (0.6) we solve the conductivity equation (0.8) for H_3 and then determine the first two components of the electric field E by formulae

$$(0.10) \quad E_1 = \frac{1}{i\omega\gamma}\partial_{x_2}H_3, \quad E_2 = -\frac{1}{i\omega\gamma}\partial_{x_1}H_3.$$

Finally, using these formulae in (0.8) to eliminate H_3 we obtain the last equation in (0.6).

Next, we claim that if the Dirichlet-to-Neuman map is given we can recover the following

A) The Dirichlet-to-Neumann map:

$$(0.11) \quad \Lambda_{1,\mu,\gamma}f = \frac{\partial u}{\partial \nu}|_{\tilde{\Gamma}},$$

where

$$(0.12) \quad L_{1,\mu,\gamma}(x, D)u = 0 \quad \text{in } \Omega, \quad u|_{\Gamma_0} = 0, \quad u|_{\tilde{\Gamma}} = f.$$

Let u be some solution to equation (0.12). We set $E_3 = u$, $q_1 = \nu_2 E_3$, $q_2 = -\nu_1 E_3$. Let $q_3 = 0$ and $H_3 = E_1 = E_2 = 0$. Finally the functions H_1 and H_2 are given by (0.9). Then

$$\vec{\nu} \times E|_{\Gamma_0} = 0$$

and

$$\begin{pmatrix} \nu_2 E_3 \\ -\nu_1 E_3 \\ \nu_1 E_2 - \nu_2 E_1 \end{pmatrix} = \mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \quad \text{on } \tilde{\Gamma}$$

and the following is known:

$$\Lambda_{\mu,\gamma}\mathbf{q} = \begin{pmatrix} \nu_2 H_3 \\ -\nu_1 H_3 \\ \nu_1 H_2 - \nu_2 H_1 \end{pmatrix} \quad \text{on } \tilde{\Gamma}.$$

The short computations imply

$$\nu_1 H_2 - \nu_2 H_1 = \frac{1}{i\omega\mu} \frac{\partial E_3}{\partial \nu} = \frac{1}{i\omega\mu} \frac{\partial u}{\partial \nu} \quad \text{on } \tilde{\Gamma}.$$

Since the trace of the function μ is known on $\tilde{\Gamma}$ we can determine $\frac{\partial u}{\partial \nu}$.

B) The Neumann-to-Dirichlet map

$$(0.13) \quad \Lambda_{2,\gamma,\mu} f = u|_{\tilde{\Gamma}},$$

where

$$(0.14) \quad L_{2,\gamma,\mu}(x, D)u = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu}|_{\Gamma_0} = 0, \quad \frac{\partial u}{\partial \nu}|_{\tilde{\Gamma}} = f.$$

Let u be some solution to equation (0.14). We set $H_3 = u$ and $E_1 = \frac{1}{i\omega\gamma} \partial_{x_2} H_3$, $E_2 = -\frac{1}{i\omega\gamma} \partial_{x_1} H_3$ and $H_1 = H_2 = E_3 \equiv 0$. Let us make a some choice of the function $\mathbf{q} = (0, 0, f)$. Then $\nu_1 E_2 - \nu_2 E_1 = -\nu_1 \frac{1}{i\omega\gamma} \partial_{x_1} H_3 - \nu_2 \frac{1}{i\omega\gamma} \partial_{x_2} H_3 = -\frac{1}{i\omega\gamma} \frac{\partial H_3}{\partial \nu} = -\frac{1}{i\omega\gamma} \frac{\partial u}{\partial \nu}$. Since the traces of ϵ and σ on $\tilde{\Gamma}$ are known the choice of f uniquely determines $\nu_1 E_2 - \nu_2 E_1$ on $\tilde{\Gamma}$. Then from the Dirichlet-to-Neumann map (0.3) we determine the traces of the functions $\nu_2 H_3$ and $\nu_1 H_3$ on $\tilde{\Gamma}$. Hence $H_3 = u$ is determined on $\tilde{\Gamma}$.

Consider the following Dirichlet-to-Neumann map

$$(0.15) \quad \Lambda_{1,\gamma,\mu} f = \frac{\partial u}{\partial \nu}|_{\tilde{\Gamma}},$$

where

$$(0.16) \quad L_{2,\gamma,\mu}(x, D)u = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu}|_{\Gamma_0} = 0, \quad u|_{\tilde{\Gamma}} = f.$$

We have

Proposition 0.1. *If the Neumann-to-Dirichlet map (0.13), (0.14) is known the Dirichlet-to-Neumann map (0.15), (0.16) is also known.*

Proof. Indeed let for some function \tilde{g} there exists a solution to the boundary value problem (0.16). We denote such a solution as u_1 . Let $\frac{\partial u_1}{\partial \nu}|_{\tilde{\Gamma}} = f$. Since $\Lambda_{2,\gamma_1,\mu_1} = \Lambda_{2,\gamma_2,\mu_2}$ there exists solution u_2 to problem (0.14) such that

$$\frac{\partial u_1}{\partial \nu}|_{\tilde{\Gamma}} = \frac{\partial u_2}{\partial \nu}|_{\tilde{\Gamma}} = f \quad \text{and} \quad u_1|_{\tilde{\Gamma}} = u_2|_{\tilde{\Gamma}} = g.$$

The proof of the Proposition is complete. ■

Making the change of unknown in (0.12) as $u = \sqrt{\mu}w$ and in (0.16) as $u = \sqrt{\gamma}w$ we have

$$\tilde{L}_{1,\mu,\gamma}(x, D)w = \Delta w + (\omega^2 \gamma \mu + \frac{\Delta \sqrt{\mu}}{\sqrt{\mu}})w = 0 \quad \text{in } \Omega$$

and

$$\tilde{L}_{2,\gamma,\mu}(x, D)w = \Delta w + (\omega^2\gamma\mu + \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}})w = 0 \quad \text{in } \Omega.$$

The above change of variables preserve the Dirichlet-to-Neumann map (0.11), (0.12) and transform the Dirichlet-to-Neumann map (0.15), (0.16) into the following one

$$(0.17) \quad \Lambda_{3,\gamma,\mu}f = \frac{\partial u}{\partial \nu}|_{\tilde{\Gamma}},$$

where

$$(0.18) \quad \tilde{L}_{2,\gamma,\mu}(x, D)u = 0 \quad \text{in } \Omega, \quad (\frac{\partial u}{\partial \nu} + au)|_{\Gamma_0} = 0, \quad u|_{\tilde{\Gamma}} = f,$$

and $a = -\frac{\partial}{\partial \nu}\sqrt{\gamma}$.

Since the potentials of the Schödinger operator can be determined from the partial Dirichlet-to-Neumann maps for the two Maxwell's systems with the same Dirichlet-to-Neumann maps we obtain

$$(0.19) \quad (\omega^2(\epsilon_1 + \frac{i\sigma_1}{\omega})\mu_1 + \frac{\Delta\sqrt{\mu_1}}{\sqrt{\mu_1}}) = (\omega^2(\epsilon_2 + \frac{i\sigma_2}{\omega})\mu_2 + \frac{\Delta\sqrt{\mu_2}}{\sqrt{\mu_2}})$$

and

$$(0.20) \quad \omega^2(\epsilon_1 + \frac{i\sigma_1}{\omega})\mu_1 + \frac{\Delta\sqrt{(\epsilon_1 + \frac{i\sigma_1}{\omega})}}{\sqrt{(\epsilon_1 + \frac{i\sigma_1}{\omega})}} = \omega^2(\epsilon_2 + \frac{i\sigma_2}{\omega})\mu_2 + \frac{\Delta\sqrt{(\epsilon_2 + \frac{i\sigma_2}{\omega})}}{\sqrt{(\epsilon_2 + \frac{i\sigma_2}{\omega})}}.$$

Denote $r = (\sqrt{\mu_1} - \sqrt{\mu_2}, \sqrt{(\epsilon_1 + \frac{i\sigma_1}{\omega})} - \sqrt{(\epsilon_2 + \frac{i\sigma_2}{\omega})})$.

By (0.19), (0.20) there exist a matrices $A = (A_1, A_2, A_3), B \in L^\infty(\Omega)$ such that

$$\Delta r + (A, \nabla r) + Br = 0 \quad \text{in } \Omega, \quad r|_{\tilde{\Gamma}} = \frac{\partial r}{\partial \nu}|_{\tilde{\Gamma}} = 0.$$

By the uniqueness theorem for the Cauchy problem for the such a system we have $r \equiv 0$. The proof of the theorem is complete. ■

REFERENCES

- [1] P. Caro, P. Ola and S. Mikko, *Inverse boundary value problem for Maxwell equations with local data* Com. P.D.E., **34**, (2009), 1425-1464.
- [2] A. P. Calderón, *On an inverse boundary value problem*, in *Seminar on Numerical Analysis and its Applications to Continuum Physics*, 65-73, Soc. Brasil. Mat., Río de Janeiro, 1980.
- [3] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology* vol. 3, Springer-Verlag, Berlin, 2000.
- [4] O. Imanuvilov, G. Uhlmann and M. Yamamoto, *The Calderón problem with partial data in two dimensions*, J. Amer. Math. Soc., **23** (2010), 655-691.
- [5] O. Imanuvilov, G. Uhlmann and M. Yamamoto, *Inverse boundary value problem by partial data for the Neumann-to-Dirichlet map in two dimensions*, arXiv:1210.1255v1
- [6] O. Imanuvilov and M. Yamamoto, *Inverse boundary value problem for the Schrödinger equation in cylindrical domain by partial boundary data*, Inverse Problems , **29** (2013), 045002.

- [7] O. Imanuvilov and M. Yamamoto, *Uniqueness for inverse boundary value problems by Dirichlet-to-Neumann map on subboundaries*, Milan J. Math., **81** (2013), 187-258.
- [8] V. Isakov, *On uniqueness in the inverse conductivity problem with local data*, Inverse Problems and Imaging, **1** (2007), 95-105.
- [9] C. Kenig and M. Salo, *The Calderón problem with partial data on manifolds and applications*, arXiv:1211.1054.
- [10] A. Nachman, *Global uniqueness for a two-dimensional inverse boundary value problem*, Ann. of Math., **143** (1996), 71–96.
- [11] P. Ola, L. Päiväranta and E. Somersalo, *An inverse boundary value problem in electrodynamics*, Duke . Math. J., **70** (1993), 617-653
- [12] J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math., **125** (1987), 153–169.